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A transformed equation for a vanishing Poisson bracket

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Abstract. For autonomous dynamical systems with two degrees of freedom we derive an equation equivalent to the equation which follows from the condition of vanishing Poisson bracket. The new version of the equation is very suitable for an easier and more efficient handling of terms of the same degree in the velocity components appearing in second integrals of motion. The new variables used are suggested in the light of the inverse problem of dynamics.

As an application we treat the problem of integrals of motion which are homogeneous polynomials in the velocity components $\dot{x}$, $\dot{y}$.

1. Introduction

Let us consider systems with two degrees of freedom. The Poisson bracket, usually written in cartesian coordinates, is the tool either (i) for checking whether an expression of the position and velocity coordinates $x, y, \dot{x}, \dot{y}$ is a second integral associated with a given Hamiltonian of a conservative system or (ii) for constructing such a second integral, if it exists. Besides, concrete criteria are known by means of which one can check whether a given function $\Phi = \Phi(x, y, \dot{x}, \dot{y})$ can stand for a second integral of motion of any dynamical system with potential function U (independent of the velocity components or even velocity dependent) which is not given in advance (Bozis and Ichtiaroglou 1987).

One might say that the problem of constructing mathematical models of integrable systems is of equal interest to that of studying chaotic systems. Very few such systems are known and, of course, only some of them are, at the same time, of physical origin. A detailed list of references on this subject, treating exclusively integrable systems of two degrees of freedom, can be found in reports of Hietarinta (1986, 1987).

On the other hand, during the last decade, the inverse problem in dynamics has received much attention and has enlightened the relation between a given potential and the totality of orbits to which this potential can give rise. For a list of references see, e.g., Bozis and Nakhla (1986). The inverse problem to which we refer is the following: 'Given a one-parameter family of curves $f(x, y) = c$ and the energy dependence function $E = E(f)$, find all potential functions $U = U(x, y)$ which can generate these orbits'. To answer this question Szebehely (1974) offered a linear, first-order, partial differential equation in U.

The aim of this paper is to write the vanishing Poisson bracket for a given dynamical system of two degrees of freedom in certain other variables coming from inverse problem considerations. The use of these variables makes the lengthy calculations involved in these sort of problems much easier to handle. Their main advantage lies

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in that they are very cooperative in treating, by means of a unique function, a number of terms, algebraic and of the same degree in the velocity components. This is a rather usual situation. Consider, for instance, a second integral of motion which is polynomial in $\dot{x}$, $\dot{y}$ of the fourth degree. Special cases of such integrals have been studied recently, by Sen (1987) among others. The existing symmetry under time reversal of the Hamiltonian requires that the integral contains only even powers in the velocity components. There are five coefficients to account for the terms of the fourth degree, three coefficients for the second-degree terms and one for the zeroth degree, in all nine coefficients, all functions of x and y. There is no essential problem with the highest-degree coefficients which are fourth-degree polynomials in x and y. However, this is not the case for the remaining $3+1=4$ coefficients, for which an overdetermined system of partial differential equations is to be solved. The new version for the vanishing Poisson bracket, suggested in this paper, replaces the problem of determining the $3+1$ functions of x and y by a problem of finding $1+1$ functions of x, y and another variable γ. With respect to this last variable γ the coefficients are polynomials and this simplifies the process.

As an application, we prove the non-existence of non-trivial second integrals which are algebraic and homogeneous in the velocity coordinates $\dot{x}$, $\dot{y}$ other than the angular momentum integral. This problem was recently considered by Thompson (1984) in a different approach.

2. The inverse problem

Consider a point Σ, of unit mass, moving in the xy plane in the autonomous field of the potential function $U=U(x, y)$. If dots denote derivatives with respect to time t, the total energy E of the point Σ is

$$E=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-U(x, y). \tag{2.1}$$

Szebehely (1974) has shown that all potential functions $U(x, y)$, which can give rise to a preassigned one-parameter family of curves

$$f(x, y)=c \tag{2.2}$$

with a preassigned dependence $E=E(f(x, y))$ of the total energy E on each member of the family, satisfy the following linear, first-order, partial differential equation in $U(x, y)$:

$$f_x U_x+f_y U_y+\frac{2(f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2)}{f_x^2+f_y^2}(E+U)=0. \tag{2.3}$$

It is understood that, even with a given dependence $E=E(f)$, infinitely many potentials can give rise to a family of curves (2.2).

Let us now introduce the notation

$$\gamma=f_y/f_x \tag{2.4}$$

and

$$\omega=\gamma\gamma_x-\gamma_y. \tag{2.5}$$

Equation (2.3) is then written as

$$U_x+\gamma U_y=\frac{2\omega}{1+\gamma^2}(E+U). \tag{2.6}$$

It is seen that the function $\gamma(x, y)$ is equally as pertinent as the function $f(x, y)$ in Szebhely's equation but, of course, not to be forgotten is that $E = E(f)$ with functions f and γ related by equation (2.4). Given the family (2.2), γ is uniquely determined. Conversely, given a function $\gamma = \gamma(x, y)$, the family of orbits is also determined from the homogeneous linear partial differential equation (2.4).

In geometrical terms, γ can be considered as a variable indicating the slope at each point of the orbit, whereas ω is a variable related to the curvature k of the orbit at the point considered; specifically it is $\omega = (1+\gamma^2)^{3/2}k$.

Functions γ and ω appearing in equation (2.6) are now to be treated as new variables, to replace the velocity components $\dot{x}, \dot{y}$. As to the position coordinates x, y they will not be altered. From equations (2.1) and (2.6) we obtain

$$\dot{x}^2 + \dot{y}^2 = (1+\gamma^2)(U_x + \gamma U_y)/\omega. \tag{2.7}$$

On the other hand, since along any orbit $f_x\dot{x} + f_y\dot{y} = 0$, we have

$$\gamma = -\dot{x}/\dot{y}. \tag{2.8}$$

At this point let us introduce, for the partial derivatives up to the second order of the function $U(x, y)$, the conventional notation

$$U_x = p \qquad U_y = q \qquad U_{xx} = r \qquad U_{xy} = s \qquad U_{yy} = t. \tag{2.9}$$

With the aid of equations (2.7) and (2.8) we then obtain

$$\dot{x} = -\varepsilon\gamma\left(\frac{p+\gamma q}{\omega}\right)^{1/2} \qquad \dot{y} = \varepsilon\left(\frac{p+\gamma q}{\omega}\right)^{1/2} \tag{2.10}$$

where $\varepsilon = \pm 1$.

Inverting (2.10) we have

$$\gamma = -\dot{x}/\dot{y} \qquad \omega = (p\dot{y} - q\dot{x})/\dot{y}^3. \tag{2.11}$$

The above transformations are to be accompanied by $x = x$, $y = y$. It is of interest to notice that the position coordinates x, y do not appear explicitly in equations (2.10) and (2.11) but only through the first-order partial derivatives p and q of the potential function $U(x, y)$.

3. The Poisson bracket

Suppose that, for a planar motion, apart from the energy integral (2.1), there exists a second integral of the motion

$$\varphi = \varphi(x, y, \dot{x}, \dot{y}). \tag{3.1}$$

The Poisson bracket $[E, \varphi]$ vanishes identically along any orbit traced by the material point Σ, i.e.

$$\varphi_x\dot{x} + \varphi_y\dot{y} + \varphi_{\dot{x}}U_x + \varphi_{\dot{y}}U_y = 0. \tag{3.2}$$

To express equation (3.2) in the new variables x, y, γ, ω we denote by $\Phi(x, y, \gamma, \omega)$ the second integral (3.1), i.e.

$$\varphi(x, y, \dot{x}, \dot{y}) = \Phi(x, y, \gamma, \omega) \tag{3.3}$$

and, in view of equations (2.11), we write

$$\varphi_x = \Phi_x + \frac{\dot{y}r - \dot{x}s}{\dot{y}^3}\Phi_\omega \tag{3.4a}$$

$$\varphi_y = \Phi_y + \frac{\dot{y}s - \dot{x}t}{\dot{y}^3}\Phi_\omega \tag{3.4b}$$

$$\varphi_{\dot{x}} = -\frac{1}{\dot{y}}\Phi_\gamma - \frac{q}{\dot{y}^3}\Phi_\omega \tag{3.4c}$$

$$\varphi_{\dot{y}} = \frac{\dot{x}}{\dot{y}^2}\Phi_\gamma + \left(-\frac{2p}{\dot{y}^3} + \frac{3\dot{x}q}{\dot{y}^4}\right)\Phi_\omega . \tag{3.4d}$$

Taking into account the expressions (2.10) and inserting (3.4*a*–*d*) into equation (3.2) we obtain, after some straightforward calculations,

$$\gamma\Phi_x - \Phi_y + \omega\Phi_\gamma + (\omega^2 L + \omega M)\Phi_\omega = 0 \tag{3.5}$$

where

$$L = \frac{3q}{p + \gamma q} \tag{3.6a}$$

and

$$M = \frac{\gamma(r - t) + (\gamma^2 - 1)s}{p + \gamma q} . \tag{3.6b}$$

Equation (3.5) is our final result. If, for a given potential function $U(x, y)$, a solution $\Phi = \Phi(x, y, \gamma, \omega)$ in closed form of equation (3.5) can be found, then the potential is integrable. The corresponding second integral $\varphi(x, y, \dot{x}, \dot{y}) =$ constant, in cartesian coordinates, is obtained immediately in view of equations (3.3) and (2.11). The advantage of replacing equation (3.2) by (3.5) is the following: all terms of the same degree in $\dot{x}, \dot{y}$ (say n) of the second integral which we try to find are factored by the same power $\omega^{-n/2}$ of the new independent variable ω. Consequently, instead of having to deal with $n+1$ coefficients (functions of x and y) we are merely left with one function of x, y and γ, of course. In addition to that it turns out that, as regards the independent variable γ, this unique function of x, y, γ is a polynomial in γ, and this is very helpful.

4. Application

We shall prove the following.

Proposition. For two-dimensional conservative systems, second integrals of motion of the form

$$\varphi = \alpha_m(x, y)\dot{x}^m + \alpha_{m-1}(x, y)\dot{x}^{m-1}\dot{y} + \cdots + \alpha_1(x, y)\dot{x}\dot{y}^{m-1} + \alpha_0(x, y)\dot{y}^m \tag{4.1}$$

(homogeneous polynomial in $\dot{x}, \dot{y}$ of degree m) correspond to potential functions of the form $U = U[\frac{1}{2}c_0(x^2 + y^2) + c_1 x + c_2 y + c^*]$ where c_0, c_1, c_2, c^* are constants.

Apart from a suitable change of coordinates this means that no integrals of the form (4.1) exist, except for those associated with the angular momentum integral. This statement was proved recently by Thompson (1984) who commented on a paper by Xanthopoulos (1984). Using Killing tensors, Thompson proved the above proposition for $m = 1, 2$ and 3 and asserted that it can be generalised for any m but 'the calculations become rather cumbersome'.

We present here, as an application of formula (3.5), a straightforward proof for any m. The reasoning goes as follows. If $\varphi(x, y, \dot{x}, \dot{y})$ in (4.1) is indeed a second integral, corresponding to the potential function $U(x, y)$ then, by virtue of the transformation (2.10), the function

$$\Phi(x, y, \gamma, \omega) = (-\varepsilon)^m (p+\gamma q)^{m/2} \omega^{-m/2} [\alpha_m \gamma^m - \alpha_{m-1}\gamma^{m-1} + \ldots + (-1)^m \alpha_0] \tag{4.2}$$

must satisfy equation (3.5). We then seek a solution of (3.5) of the form

$$\Phi(x, y, \gamma, \omega) = (p+\gamma q)^{m/2} \omega^{-m/2} F(x, y, \gamma). \tag{4.3}$$

Inserting the expression (4.3) into (3.5) we observed that only two powers $\omega^{-m/2}$ and $\omega^{-m/2+1}$ of the independent variable appear and this readily leads to the following two equations for the unique unknown function $F = F(x, y, \gamma)$:

$$\gamma F_x - F_y = 0 \tag{4.4a}$$

and

$$(p+\gamma q) F_\gamma = mqF. \tag{4.4b}$$

The general solution of equation (4.4b) is

$$F(x, y, \gamma) = (p+\gamma q)^m g(x, y) \tag{4.5}$$

where $g(x, y)$ is an arbitrary function of its arguments x, y. Inserting the solution into equation (4.4a) and arranging the result in powers of γ, we obtain

$$(msg + qg_x)\gamma^2 + [(mrg + pg_x) - (mtg + qg_y)]\gamma - (msg + pg_y) = 0. \tag{4.6}$$

Since γ is an independent variable, the function $g = g(x, y)$ must satisfy the system of equations

$$-\frac{1}{m}\frac{g_x}{g} = \frac{s}{q} \tag{4.7a}$$

$$mrg + pg_x = mtg + qg_y \tag{4.7b}$$

$$-\frac{1}{m}\frac{g_y}{g} = \frac{s}{p}. \tag{4.7c}$$

Since $U(x, y)$ depends both on x and y, both p and q are different from zero. In fact, equation (4.7b) can also be written, in view of equations (4.7a) and (4.7c), as

$$\frac{p^2 - q^2}{pq} s = r - t. \tag{4.8}$$

On the other hand, equations (4.7a) and (4.7c) are compatible if and only if

$$(s/q)_y = (s/p)_x. \tag{4.9}$$

The general solution of this last equation is

$$U = U(z) \tag{4.10}$$

with

$$z = \alpha(x) + \beta(y) \tag{4.11}$$

and $\alpha(x)$, $\beta(y)$ arbitrary functions of x and y, respectively. The solution of equations (4.7a) and (4.7c) for g is

$$g(x, y) = c^m / U_z^m \tag{4.12}$$

where c is a constant and $U_z \neq 0$.

We now demand that the solution (4.10) also satisfies equation (4.8) (which has replaced equation (4.7b)) and this leads directly to the equation

$$(\alpha'' - \beta'') U_z = 0 \tag{4.13}$$

where primes in α and β denote differentiation with respect to x and y, respectively. The conclusion is that

$$\alpha'' = \beta'' = c_0 \tag{4.14}$$

i.e.

$$\alpha(x) = \tfrac{1}{2} c_0 x^2 + c_1 x + c_3 \tag{4.15a}$$

$$\beta(y) = \tfrac{1}{2} c_0 y^2 + c_2 y + c_4 \tag{4.15b}$$

where c_0, c_1, c_2, c_3, c_4 are all constants. Therefore

$$U = U[\tfrac{1}{2} c_0 (x^2 + y^2) + c_1 x + c_2 y + c^*] \tag{4.16}$$

with $c^* = c_3 + c_4$, i.e. U is the potential function of a central force field, centred, for $c_0 \neq 0$, at the point $(-c_1/c_0, -c_2/c_0)$ and this completes the proof of the proposition.

The integral $\varphi(x, y, \dot{x}, \dot{y}) =$ constant itself can be found directly as follows. Since $\Phi(x, y, \gamma, \omega) =$ constant, from equations (4.3), (4.5) and (4.12) we obtain

$$(p + \gamma q)^{3/2} / \omega^{1/2} U_z = \text{constant}. \tag{4.17}$$

Taking into account equations (2.9), (2.11) and also (4.10) and (4.11), we write (4.17) as $\dot{y}\alpha' - \dot{x}\beta' =$ constant, or, finally, in view of equations (4.15a, b), as

$$c_0(x\dot{y} - \dot{x}y) + c_1 \dot{y} - c_2 \dot{x} = \text{constant} \tag{4.18}$$

and this expresses the constancy of the angular momentum integral with respect to the point $(-c_1/c_0, -c_2/c_0)$.

For $c_0 = 0$, equation (4.18) expresses the constancy of the linear momentum along a line in the xy plane.

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